Chapter 2

Differential Geometry

Abstract of this chapter. Stichwort: Lokale Maße.

Todo in Chapter 2: Check consistent usage of vecThree and vecThree-Transp. Also check that operators in line wrapped equations do not appear on the end of the line.

2.1 Curves

The curves we consider in this lecture can be regarded as a continuous function mapping some one dimensional real interval into three dimensional space:

$$\mathbf{f} : [a, b] \to \mathbb{R}^3, \quad a, b \in \mathbb{R} \text{ and } a < b$$

$$\mathbf{f}(t) = (x(t), y(t), z(t))^\mathsf{T}$$

$\mathbf{f}(t)$ is a so called *parametrization* of the curve (see figure 2.1).

For the sake of simplicity we will oftentimes use two dimensional curves in these lecture notes. Most of the important concepts, however, apply to both, two and three dimensional curves.

2.1.1 Arc Length Parametrization

A specific curve c is geometrically uniquely defined by the set of points that lie on it. This set of points can be described as

$$c = \{p \mid \exists t \in [a, b] \text{ with } p = \mathbf{f}(t)\} .$$

Observe, however, that there are infinitely many $\mathbf{f}(t)$ which describe the same set c. So in essence, we have infinitely many parametrizations $\mathbf{f}$ of the curve we can choose from. Now, we want to introduce a specific parametrization, the so called *arc length parametrization* which has some favorable properties, namely:

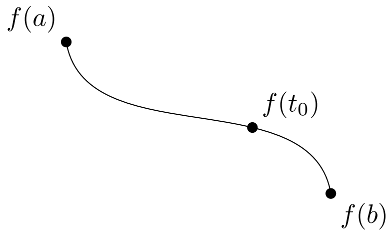

Figure 2.1: A sample curve in $\mathbb{R}^2$ with some arbitrary point $\mathbf{f}(t_0)$ highlighted.

- The distance of two points t_0 and t_1 in parameter space is the same as the distance along the curve between the mapped points $\mathbf{f}(t_0)$ and $\mathbf{f}(t_1)$. Or, more informally: The distance we travel in parameter space is the same as the distance we travel along the curve. See figure 2.2.
- The first order derivative $\mathrm{d}/\mathrm{d}t\,\mathbf{f}$ is orthogonal to the second order derivative $\mathrm{d}^2/\mathrm{d}t^2\,\mathbf{f}$.

A curve $\mathbf{f} : [a, b] \to \mathbb{R}^3$ can be reparametrized by a continuous function $t(s) : [a', b'] \to [a, b]$ with

$$\tilde{\mathbf{f}}(s) := \mathbf{f}(t(s))$$

such that $\tilde{\mathbf{f}}([a', b']) = \mathbf{f}([a, b])$ i.e. $\tilde{\mathbf{f}}$ and $\mathbf{f}$ are representations of the same curve. t is called a *reparametrization* of $\mathbf{f}$.

$\tilde{\mathbf{f}}$ can be derived using the chain rule:

$$\frac{\mathrm{d}}{\mathrm{d}s}\tilde{\mathbf{f}}(s) = \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{f}(t(s)) \cdot \frac{\mathrm{d}t}{\mathrm{d}s} \ . \tag{2.1}$$

The arc length of the curve can be determined with the integral over the length of the derivative:

$$L = \int_a^b \left\| \frac{\mathrm{d}\mathbf{f}(t)}{\mathrm{d}t} \right\| \mathrm{d}t = \int_{a'}^{b'} \left\| \frac{\mathrm{d}\tilde{\mathbf{f}}(s)}{\mathrm{d}s} \right\| \mathrm{d}s$$

with $t(a') = a$ and $t(b') = b$. Applying equation 2.1 we can substitute:

$$L = \int_{a'}^{b'} \left\| \frac{\mathrm{d}\mathbf{f}(t(s))}{\mathrm{d}t} \right\| \frac{\mathrm{d}t}{\mathrm{d}s}\mathrm{d}s = \int_{a'}^{b'} \left\| \frac{\mathrm{d}\tilde{\mathbf{f}}(s)}{\mathrm{d}s} \right\| \mathrm{d}s$$

A curve $\bar{\mathbf{f}}$ is arc length parametrized if $\left\| \frac{\mathrm{d}}{\mathrm{d}s}\bar{\mathbf{f}}(s) \right\| = 1$. To yield an arc length parametrized representation of an arbitrarily parametrized curve $\mathbf{f}$ we define

$$s(t) := \int_a^t \left\| \frac{\mathrm{d}\mathbf{f}(u)}{\mathrm{d}u} \right\| \mathrm{d}u \ . \tag{2.2}$$

The inverse function $t(s) := s^{-1}(t)$ is called the *arc length parametrization* of $\mathbf{f}$. Plugging $t(s)$ into $\mathbf{f}$ we yield $\bar{\mathbf{f}}(s) = \mathbf{f}(t(s))$, the arc length parameterized representation of $\mathbf{f}$.

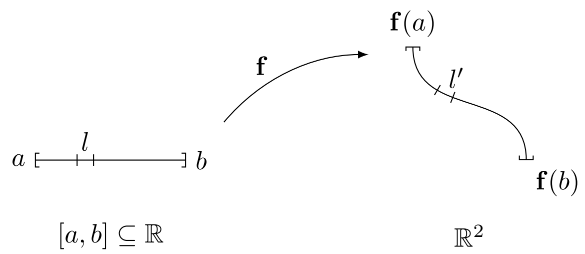

Figure 2.2: An arc-length parametrized curve is length preserving, i.e. $l = l'$.

Proof. From equation 2.2 and the fundamental theorem of calculus follows:

$$\frac{\mathrm{d}s(t)}{\mathrm{d}t} = \left\| \frac{\mathrm{d}\mathbf{f}(t)}{\mathrm{d}t} \right\| .$$

Because of the Inverse Function Theorem the following equation holds:

$$\frac{\mathrm{d}t(s)}{\mathrm{d}s} = \frac{1}{\frac{\mathrm{d}s(t)}{\mathrm{d}t}} = \frac{1}{\left\| \frac{\mathrm{d}\mathbf{f}(t)}{\mathrm{d}t} \right\|} \tag{2.3}$$

Applying the last two equations to the arc-length parameterized curve function, it turns out that the length of the derivative is indeed always equal to 1.

$$\begin{aligned} \left\| \frac{\mathrm{d}\bar{\mathbf{f}}(s)}{\mathrm{d}s} \right\| &\overset{(2.2)}{=} \left\| \frac{\mathrm{d}\mathbf{f}(t)}{\mathrm{d}t} \cdot \frac{\mathrm{d}t}{\mathrm{d}s} \right\| = \left\| \frac{\mathrm{d}\mathbf{f}(t)}{\mathrm{d}t} \right\| \cdot \left\| \frac{\mathrm{d}t}{\mathrm{d}s} \right\| \\ &\overset{(2.3)}{=} \left\| \frac{\mathrm{d}\mathbf{f}(t)}{\mathrm{d}t} \right\| \cdot \frac{1}{\left\| \frac{\mathrm{d}\mathbf{f}(t)}{\mathrm{d}t} \right\|} = 1 \end{aligned}$$

$\square$

From the fact that $\left\| \mathrm{d}/\mathrm{d}s\, \bar{\mathbf{f}}(s) \right\| = 1$ we can now immediately derive that $\mathrm{d}/\mathrm{d}s\, \bar{\mathbf{f}}(s) \perp \mathrm{d}^2/\mathrm{d}s^2\, \bar{\mathbf{f}}(s)$:

$$\left\| \frac{\mathrm{d}\bar{\mathbf{f}}(s)}{\mathrm{d}s} \right\|^2 = 1^2 \overset{\mathrm{d}/\mathrm{d}s}{\Leftrightarrow} 2 \left\| \frac{\mathrm{d}\bar{\mathbf{f}}(s)}{\mathrm{d}s} \frac{\mathrm{d}^2\bar{\mathbf{f}}(s)}{\mathrm{d}s^2} \right\| = 0 \Leftrightarrow \frac{\mathrm{d}\bar{\mathbf{f}}(s)}{\mathrm{d}s} \perp \frac{\mathrm{d}^2\bar{\mathbf{f}}(s)}{\mathrm{d}s^2}$$

2.1.2 Frenet Frame

Now, a valuable tool for analyzing certain local properties of a curve, the *Frenet Frame*, will be introduced. The Frenet Frame is a natural local coordinate system defined at a specific point on the curve.

Assuming that the first three derivatives of a curve $\mathbf{f}(t)$ exist, the first terms of its Taylor expansion are given by

$$\begin{aligned} \mathbf{f}(t) &= \mathbf{f}(0) + t\mathbf{f}'(t) + \frac{1}{2}t^2\mathbf{f}''(t) + \frac{1}{6}t^3\mathbf{f}'''(t) + \ldots \\ &= \mathbf{f}(0) + \begin{pmatrix} \mathbf{f}'(t) & \mathbf{f}''(t) & \mathbf{f}'''(t) \end{pmatrix} \begin{pmatrix} t \\ t^2/2 \\ t^3/6 \end{pmatrix} + \ldots . \end{aligned}$$